About the Kernel of the Augmentation of finitely generated Z-modules.

by

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Abstract: Let M be a free finitely generated $\mathbf{Z}$-module with basis B and ΔM the kernel of the homomorphism $M \to \mathbf{Z}$ which maps B to 1. A basis of ΔM can be easily constructed from the basis B of M. Let further R be a submodule of M such that $N = M/R$ is free. Subject of investigation is the module $\Delta N = (\Delta M + R)/R$. We compute the index $[N : \Delta N]$ and construct bases of ΔN with the help of a basis of N. Finally the results are applied to a special class of modules which is connected with the group of cyclotomic units.

1 Introduction

Well known in the context of group rings is the *augmentation* of an group ring element which is the homomorphism obtained by mapping the group elements to 1. The augnmentation defines the *augmentation ideal* of the group ring which denotes the kernel of the augmentation [3]. Similarly, in a free $\mathbf{Z}$-module M each basis B defines an homomorphism $\mathrm{aug} : M \to \mathbf{Z}$, $\sum_{b\in B} \alpha_b b \mapsto \sum_{b\in B} \alpha_b$. We denote the kernel of aug by ΔM. We consider further the module $N = M/R$ where R is a submodule of M such that N is free, and let $\Delta N = (\Delta M + R)/R$. In the following we assume that the module M is finitely generated. It is easy to see that the index $[M : \Delta M]$ is infinite. In Theorem 2.1 we identify the index $[N : \Delta N]$ as the greatest common divisor of the augmentation of the elements of R.

It can be seen straightforwardly that for a fixed $b_0 \in B$ the set

$$B_0 = \{b - b_0;\ b \in B,\ b \neq b_0\} \tag{1}$$

is a basis of ΔM. A similar result is obtained for ΔN in Theorem 3.2. In Section 4 we will apply this result to a class of modules which is connected to the group of cyclotomic units. This group plays an important role in the theory of cyclotomic fields [4].

2 The index of ΔN

We use in the following the notation of the introduction.

Theorem 2.1 *We have* $[N : \Delta N] = \gcd \operatorname{aug}(R)$ *where the greatest common divisor of* $\{0\}$ *is defined as* ∞.

Proof: Let $b_0 \in B$. Because $b \equiv b_0 \bmod \Delta M$ for all $b \in B$ we see that $N/\Delta N$ is cyclic and generated by b_0. The index is the smallest positive number such that $mb_0 \in R + \Delta M$. Note that $\operatorname{aug}(mb_0) = m$ for $m \in \mathbf{Z}$.
In the case $R \subseteq \Delta M$ we have $\operatorname{aug} R = \{0\}$. From $mb_0 \notin \Delta M$ for all $m \neq 0$ we see $[N : \Delta N] = \infty$ as it was claimed in the Theorem.
For $R \not\subseteq \Delta M$ there exists an element $r \in R$ with minimal positive augmentation ρ. Noting that $\rho b_0 \equiv r \bmod \Delta M$ it follows $[N : \Delta N] \leq \rho$.
If we have on the other hand $k \in \mathbf{Z}$ and $r' \in R$ such that $kb_0 \equiv r' \bmod \Delta M$ it follows $\rho \leq k = \operatorname{aug}(r')$ because of the minimality of ρ, and we obtain $[N : \Delta N] = \rho$.
It remains to show that $\rho = \gcd \operatorname{aug}(R)$. Suppose there exists $r' \in R$ such that ρ is not a divisor of $\rho' = \operatorname{aug}(r')$. Then we find by computing $\delta = \gcd(\rho, \rho')$ numbers $\alpha, \beta \in \mathbf{Z}$ with $\delta = \alpha\rho + \beta\rho'$. But $\alpha r + \beta r' \in R$ is an element with positive augmentation $\delta < \rho$ which is a contradiction to the minimality of ρ. □

We show in the next lemma how the index $[N : \Delta N]$ can be explicitly computed.

Lemma 2.2 *If* $E \subseteq R$ *generates* R *then* $\gcd \operatorname{aug}(R) = \gcd \operatorname{aug}(E)$.

Proof: For $[N : \Delta N] = \infty$ there is nothing to show. In the case that $\rho = [N : \Delta N] < \infty$ the claim follows from the existence of $r \in R$ and $\alpha_e \in \mathbf{Z}$ such that $\rho = \operatorname{aug}(r) = \sum_{e \in E} \alpha_e \operatorname{aug}(e)$. With this we obtain

$$\gcd \operatorname{aug}(R) = \rho = \gcd(\operatorname{aug}(E) \cup \{\rho\}) = \gcd \operatorname{aug}(E). \tag{2}$$

□

Remark 2.3 *Similarly to ΔM we can identify ΔN as a kernel of an homomorphism. With $k = [N : \Delta N]$ for finite and $k = 0$ for infinite index we have an homomorphism*

$$\overline{\operatorname{aug}} : N \to \mathbf{Z}/k\mathbf{Z},\ a + R \mapsto \operatorname{aug}(a) + k\mathbf{Z} \tag{3}$$

and $\Delta N = \ker_N \overline{\operatorname{aug}}$.

3 Construction of a Basis of ΔN

In the following, let $C \subseteq M$ induce a basis of N, i. e. let $\{c + R;\ c \in C\}$ be a basis of N. We assume that there exist $\gamma \in \mathbf{Z}$ such that $\operatorname{aug}(c) = \gamma$ for all $c \in C$. Note that this is no restriction to the module N. In Algorithm 3.4 we will show how such a basis can be constructed from an arbitrary basis of N.

Let $\rho = [N : \Delta N]$. In the case $\rho = \infty$ it is easy to see that similarly to (1) for a fixed $c_0 \in C$ the set $C_0 = \{c - c_0;\ c \in C,\ c \neq c_0\}$ is a basis of ΔN. We assume in the following $\rho < \infty$ and show in the next Lemma and the subsequent Theorem how to construct bases of ΔN in this case.

Lemma 3.1 *Let $c_1 \in C$. Then*

$$C_1 = \{c - c_1;\ c \in C, c \neq c_1\} \cup \{\rho c_1\} \tag{4}$$

induces a basis of ΔN.

Proof: We show that the elements $b - b_0$ with $b, b_0 \in B$ are modulo R generated by C_1. Because C is a basis of N we have $\alpha_c, \beta \in \mathbf{Z}$ such that

$$b - b_0 = r + \beta c_1 + \sum_{c \in C,\, c \neq c_1} \alpha_c (c - c_1). \tag{5}$$

The application of aug to (5) and reducing modulo ρ gives $\beta\gamma \equiv 0 \bmod \rho$. We show in the rest of the proof that $\gcd(\gamma, \rho) = 1$. Then we have $\rho | \beta$ and the claim of the Lemma follows.

We can write any $b \in B$ as $b = c + r$ with $c \in \langle C \rangle$ and $r \in R$. This gives $1 = \mathrm{aug}(c) + \mathrm{aug}(r) = \nu\gamma + \mu\rho$ with $\nu, \mu \in \mathbf{Z}$ which leads to $\gcd(\gamma, \rho) = 1$. □

Compared with the basis B_0 of ΔM in (1) the basis C_1 from (4) has the extra element ρc_1 added to the expected elements $c - c_1$. In the next theorem we give a basis which looks more similar to B_0.

Theorem 3.2 *Let* $c_0 \equiv c' \bmod R$ *such that* $c' \in \langle C \rangle$ *and* $\mathrm{aug}(c') = (1-\rho)\gamma$. *Then*

$$C_0 = \{c - c_0;\ c \in C\} \tag{6}$$

induces a basis of ΔN.

Proof: Let c_1 as in Lemma 3.1 and $C' = \{c - c_1;\ c \in C,\ c \neq c_1\}$ such that $C_1 = C' \cup \{\rho c_1\}$ induces a basis of ΔN. Because of $c_1 - c' \equiv \rho c_1 \bmod \langle C' \rangle$ we can replace ρc_1 by $c_1 - c'$ in C_1. By replacing the other elements of C_1 using the relation $c - c' = c - c_1 + (c_1 - c')$ for $c \in C$ we obtain $\{c - c';\ c \in C\}$ as a basis of ΔN. With $c_0 \equiv c' \bmod R$ we get the claim. □

Remark 3.3 *If we choose* $c_0 = c' + \gamma r$ *with* $r \in R$ *such that* $\mathrm{aug}(r) = \rho$ *we obtain* $\mathrm{aug}(c_0) = \gamma$ *and therefore* $C_0 \subseteq \Delta M$. *So, with* C_0*, we obtain directly a basis of* $\Delta M/(\Delta M \cap R)$ *(which is of course isomorphic to* ΔN*).*

In Lemma 3.1 and Theorem 3.2 we assume that there is a basis $C \subseteq M$ of N with $\mathrm{aug}(c) = \gamma$ for all $c \in C$. We give here an Euclidean-like algorithm which shows how to construct such a basis from an arbitrary basis.

Algorithm 3.4 *Let $C \subseteq M$ induce a basis of N. The algorithm leads to* $\operatorname{aug}(c) = \gamma$ *for all $c \in C$ by successively replacing elements of C.*

If $\operatorname{aug}(c) = 0$ *for all $c \in C$ it remains nothing to do. Otherwise we chose first c' with* $\operatorname{aug}(c') \neq 0$ *and replace each $c \in C \setminus \{c'\}$ by $c + \lambda c'$ with $\lambda \in \mathbf{Z}$ such that* $\operatorname{aug}(c) > 0$. *If* $\operatorname{aug}(c') < 0$ *we have also to replace c' by $-c'$. After that we perform the following steps.*

1. *If all elements of C have the same augmentation the algorithm is finished.*
2. *Pick c, $c' \in C$ such that* $\operatorname{aug}(c) < \operatorname{aug}(c')$ *and replace c' by $c' - c$.*
3. *Go to Step 1.*

The algorithm terminates because $\sum_{c \in C} \operatorname{aug}(c) \in \mathbf{N}$ *decreases in every run of Step 2.*

4 A special class of modules

For a finite set A we denote by ΣA the sum $\sum_{a \in A} a$ in the free module $\langle A \rangle$ generated by A. For $i = 1, \ldots, r$ let A_i be a finite set with an involution σ operating nontrivially on each element. So we have sets H_i such that $B_i = H_i \cup \sigma H_i$ and $H_i \cap \sigma H_i = \emptyset$ for $i = 1, \ldots, r$. We define further the module

$$Z = \langle B_1 \rangle / \langle \Sigma B_1 \rangle \otimes \cdots \otimes \langle B_r \rangle / \langle \Sigma B_r \rangle. \tag{7}$$

The involution on the B_i defines an involution on Z and we may interpret Z also as a $\mathbf{Z}[\sigma]$-module. Subject of investigation is the module $N = Z / \ker_Z(\sigma + 1)$.

Remark 4.1 *The module N is directly connected with the group of cyclotomic units $C^{(n)}$. Let ϵ_n be a primitive nth root of unity. Then $C^{(n)}$ is defined* *as the multiplicative subgroup of* $D^{(n)}$ *which are units of* $\mathbf{Z}[\epsilon_n]$. *The group* $D^{(n)}$ *is generated by the elements* $1-\epsilon_n^a$ *with* $1 \le a < n$ *modulo torsion. With* $\widehat{C^{(n)}} = C^{(n)}/L^{(n)}$ *where* $L^{(n)} = \prod_{d|n,\, d\neq n} C^{(d)}$ *we have for* $n = p_1 \cdots p_r$ *odd, square free and not a prime an isomorphism* $N \cong \widehat{C^{(n)}}$ *when we choose* $B_i = \{1, \ldots, p_i - 1\}$. *For general* n *we have similar isomorphisms (see [1]).*

Let $M = \langle B_1 \times \cdots \times B_r \rangle$ and let S be the module generated by the sums

$$s_i(a_1, \ldots, a_r) = \sum_{b \in B_i} (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_r), \quad i = 1, \ldots, r \tag{8}$$

where $a_j \in B_j$ for $j = 1, \ldots, r$. By [2] we have then for r even

$$N \cong M/(S + (1-\sigma)M) \tag{9}$$

and for r odd

$$N \cong M/(S + (1-\sigma)M + \langle e \rangle) \tag{10}$$

with $e = \Sigma(H_1 \times \cdots \times H_r)$.

Theorem 4.2 *For* $i = 1, \ldots, r$ *let* $\beta_i = |B_i|$, *the number of elements of* B_i, *and* $\beta = \gcd(\beta_1, \ldots, \beta_r)$. *Then we have for* $\rho = [N : \Delta N]$ *that*

$$\rho = \begin{cases} \beta/2 & \text{if } r = 1 \text{ or} \\ & r \text{ odd and } \beta_i \equiv 2 \bmod 4 \text{ for } i = 1, \ldots, r, \\ \beta & \text{else.} \end{cases}$$

Proof: The claim follows for r even and $r = 1$ from Lemma 2.2 and the isomorphisms (9) and (10). For r odd we additionally use $\operatorname{aug}(e) = \prod_{i=1}^r (\beta_i/2)$. □

A basis of N can be constructed with weak σ-bases according to [1]. We get the following result.

Lemma 4.3 *For each* $i = 1, \ldots, r$ *we fix* $h_i \in H_i$. *Let* $H_i^\flat = H_i \setminus \{h_i\}$ *and* $A_i^\flat = A_i \setminus \{h_i\}$. *Then we obtain* $C = F^0 \cup F^+$ *as a basis of* N *where*

$$F^0 = \bigcup_{i=1}^r \{h_1\} \times \cdots \times \{h_{i-1}\} \times H_i^\flat \times B_{i+1}^\flat \times \cdots \times B_r^\flat \tag{11}$$

and

$$F^+ = \begin{cases} \emptyset & \text{for } r \text{ odd,} \\ \{h_1\} \times \cdots \times \{h_r\} & \text{for } r \text{ even.} \end{cases} \tag{12}$$

We see here that the basis C can be chosen as a subset of $B = B_1 \times \cdots \times B_r$. So, all elements of C have augmentation 1, and we may apply Theorem 3.2 with $\gamma = 1$ and $c_0 = (1-\rho)c$ where c is any element of C. This leads to a basis of ΔN as in (6).

However, we might ask a stronger question: Can we find a basis $C \subseteq B$ of N such that we can choose $c_0 \in B$? Up to now there is no general answer to this. We will discuss in the rest of this section some special cases where the answer is affirmative.

In the following, we call a basis C_0 of ΔN which has the form $C_0 = \{c - c_0;\ c \in C\}$ with $C \subseteq B$ and $c_0 \in B$ a *handsome* basis of ΔN.

Theorem 4.4 *If there exists a $j \in \{1, \ldots, r\}$ such that $\beta_j = \rho$ then ΔN has a handsome basis.*

Proof: In the case $F_0 \neq \emptyset$ we rearrange the sets B_i such that $j = r$. Let $(a_1, \ldots, a_{r-1}, a_r)$ be any element of F^0. Then $(a_1, \ldots, a_{r-1}, b) \in F^0$ for $b \neq h_r$. So we may choose in Theorem 3.2 $c' = -\sum_{b \in B^\flat}(a_1, \ldots, a_{r-1}, b)$, and the claim follows with $c_0 = (a_1, \ldots, a_{r-1}, h_r)$. In the case $F_0 = \emptyset$ we take $c_0 = (h_1, \ldots, h_{r-1}, \sigma h_r)$. □

The converse of Theorem 3.2 is not true. Even if a basis of the form $C = F^0 \cup F^+$ as in Lemma 4.3 cannot be used for the construction of a handsome basis we may have more success when starting with a different basis. We will give an example in the next Lemma.

Lemma 4.5 *Let $B_1 = \{a, b, \sigma a, \sigma b\}$ and $B_2 = \{a, b, c, \sigma a, \sigma b, \sigma c\}$ two sets of four respectively six elements with σ acting nontrivially on B_1 and B_2. The module N is as in (9) given as the free module M generated over $B = B_1 \times B_2$ modulo $(1-\sigma)M$ and the relations described in (8). With*

$$C = \{(a, b), (b, c), (\sigma b, a), (\sigma b, b), (\sigma b, c), (a, \sigma a), (a, \sigma b), (\sigma a, c)\}$$

and $c_0 = (a, a)$ *we obtain* $\{c - c_0;\ c \in C\}$ *as a basis of* ΔN.

Proof: We show first that C is a basis of N. By Lemma 4.3 we obtain $\operatorname{rank} N = 8 = |C|$ and it is sufficient to show that C generates N. Because of $c \equiv \sigma c \bmod (1-\sigma)M$ we see that σC is generated by C. The elements of $B \setminus (C \cup \sigma C)$ can then be generated by $C \cup \sigma C$ directly by relations of S. Theorem 2.1 gives $[N : \Delta N] = 2$. We will now apply Theorem 3.2 in order to construct a basis of ΔN. Let $c' = c_0 + r$ with

$$r = \sum_{x \in B_1} (x, c) - \sum_{y \in B_2} (a, y) - (1-\sigma)(\sigma a, c). \tag{13}$$

Because c' satisfies the conditions in Theorem 3.2 the claim follows. □

Without going into details we note that a construction as in Lemma 4.5 can be generalized to more complicated cases. However, the problem to give a general algorithm for the construction of a handsome basis remains open.

References

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