

— Weak σ -bases —

Let M be a module with an involution σ .

A **weak σ -basis** of M is a triple $[E^0, E^+, E^-]$ of subsets of M such that the union

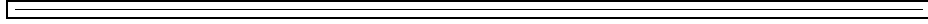
$$B = E^0 \cup \sigma E^0 \cup E^+ \cup E^-$$

is disjoint, B is a basis of M and

$$\sigma e \equiv e \pmod{\langle E^0 \cup \sigma E^0 \rangle} \text{ for } e \in E^+,$$

$$\sigma e \equiv -e \pmod{\langle E^0 \cup \sigma E^0 \rangle} \text{ for } e \in E^-.$$

We write $B = [E^0, E^+, E^-]$ for short.



Note that

$$m^+ = m^+(M) = |E^+| \text{ and } m^- = m^-(M) = |E^-|$$

are invariants of M . We have

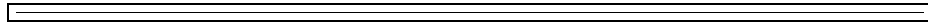
$$H^0(\sigma, M) \cong \mathbf{F}_2^{m^+} \text{ and } H^1(\sigma, M) \cong \mathbf{F}_2^{m^-}.$$

— Examples —

$$A = \{a, \sigma a, b, \sigma b\}, \mathcal{E} = \left\{ \sum_{x \in A} x \right\}:$$

$[\{a, b\}, \emptyset, \emptyset]$ defines a weak σ -basis of $\langle A \rangle$,

$[\{a\}, \emptyset, \{b\}]$ defines a weak σ -basis of $\langle A \rangle / \langle \mathcal{E} \rangle$.



Let

$B = [E^0, E^+, E^-]$ be a weak σ -basis of M .

$C = [F^0, F^+, F^-]$ be a weak σ -basis of L .

Then $[G^0, G^+, G^-] \subseteq M \times L$ with

$$G^0 = (E^0 \times C) \cup (E^+ \times F^0) \cup (E^- \times F^0),$$

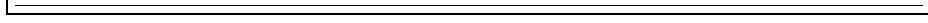
$$G^+ = (E^+ \times F^+) \cup (E^- \times F^-),$$

$$G^- = (E^+ \times F^-) \cup (E^- \times F^+)$$

defines a weak σ -basis of $M \otimes L$.

— Exact Sequences —

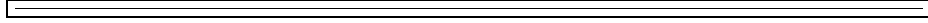
Let $[E^0, E^+, E^-]$ be a weak σ -basis of M . Then $E^0 \cup E^+$ defines a basis of $M_+ = M/\ker_M(\sigma + 1)$.



Lemma 1 *Given an exact sequence*

$$0 \rightarrow M \rightarrow L \rightarrow K \rightarrow 0. \quad (*)$$

Let $[F^0, F^+, F^-] \subseteq L$ define a weak σ -basis of K . If $()$ splits over σ then $E^0 \cup E^+ \cup F^0 \cup F^+$ defines a basis of L_+ .*



Lemma 2 *Let*

$$0 = L^{(0)} \leq L^{(1)} \leq \dots \leq L^{(i)} \leq \dots \leq L = \bigcup_{i=0}^{\infty} L^{(i)}.$$

be a chain with the property that for every $i \in \mathbf{N}$ there exists a module $M^{(i)}$ such that the sequence

$$0 \rightarrow L^{(i-1)} \rightarrow L^{(i)} \rightarrow M^{(i)} \rightarrow 0$$

is exact and splits over σ . If $B_+^{(i)} \subseteq L^{(i)}$ defines a basis of $M_+^{(i)}$ for all $i \in \mathbf{N}$ then $\bigcup_{i=1}^{\infty} B_+^{(i)}$ defines a basis of L_+ .

Let Δ be an appropriate indexing set and for $d \in \Delta$:

M_d a module,

$\mathcal{E}_d \subseteq M_d$,

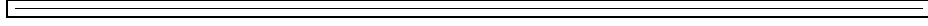
$\mathbf{n}_d : \mathcal{E}_d \rightarrow \bigoplus_{t < d} M_t$ a mapping.

Then we call the module $\mathcal{L} = N/Q$ with

$$N = \bigoplus_{t \in \Delta} M_t,$$

$$Q = \sum_{t \in \Delta} \langle r + \mathbf{n}_t(r); r \in \mathcal{E}_t \rangle$$

the **combination** of the system $\Gamma = (M_d, \mathcal{E}_d, \mathbf{n}_d)_{d \in \Delta}$.



Theorem 1 *If Γ is combinable and splits over σ (two technical conditions) we have:*

If $B_+^{(d)} \subseteq M_d$ defines a basis of $(M_d/\langle \mathcal{E}_d \rangle)_+$ for each $d \in \Delta$ then $\bigcup_{d \in \Delta} B_+^{(d)} \subseteq$

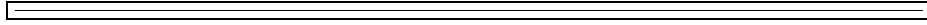
$\bigoplus_{d \in \Delta} M_d$ defines a basis of \mathcal{L}_+ .

Let

$$G_d = \{1 \leq b < d; (b, d) = 1\}, \sigma b = d - b \text{ for } b \in G_d,$$

$$A_p = \{0, \dots, p-1\}, \quad \sigma a = p-1-a \text{ for } a \in A_p.$$

Write $\Sigma(S)$ for $\sum_{s \in S} s$.



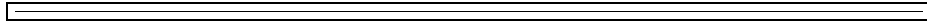
Define the **cyclotomic module** $Z(n)$ as follows:

$$\text{For } n = p \text{ prime let } Z(p) = \langle G_p \rangle / \langle \Sigma(G_p) \rangle.$$

For $n = q = p^\alpha$, $\alpha > 1$ let

$$Z(q) = \langle G_{q/p} \rangle \otimes \langle A_p \rangle / \langle \Sigma(A_p) \rangle.$$

For $n = q_1 \cdots q_r$ let $Z(n) = Z(q_1) \otimes \cdots \otimes Z(q_r)$.



Lemma 3

$$Z(n) \cong M_n / \langle \mathcal{E}_n \rangle$$

where $M_n = \langle G_n \rangle$ and

$$\mathcal{E}_n = \{s(n, p, a); p|n \text{ with } p \text{ prime, } a \in G_{n/p}\}$$

with

$$s(n, p, a) = \Sigma(\{x \in G_n; x \equiv a \pmod{(n/p)}\}).$$

The n th cyclotomic system $\Gamma(n)$ is defined as a system $(M_d, \mathcal{E}_d, \mathbf{n}_d)_{d|n}$ with

$$M_d = \langle G_d \rangle,$$

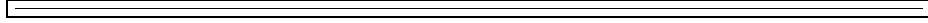
\mathcal{E}_d as before if d is not prime, else $\mathcal{E}_d = \emptyset$,

$$\mathbf{n}_d : \mathcal{E}_d \rightarrow \bigoplus_{t|d, t \neq d} M_t$$

$$s(d, p, a) \mapsto \begin{cases} -[d/p; a] & \text{if } p^2 | d, \\ [d/p; p^{-1}a] - [d/p; a] & \text{if } p^2 \nmid d, \end{cases}$$

where $[m; x]$ means $y \in G_m$ with $x \equiv y \pmod{m}$.

We denote the combination of $\Gamma(n)$ by $\mathcal{L}(n)$.



Lemma 4

If $4 \nmid n$ then $\Gamma(n)$ is combinable and splits over σ .

If $4|n$ we can make some modifications to get a similar result.

\longrightarrow *we can construct a basis of $\mathcal{L}(n)_+$ by weak σ -bases of the modules $M_d/\langle \mathcal{E}_d \rangle$.*

Let ϵ_d be a primitive d th root of unity. We call

$$D^{(n)} = \langle 1 - \epsilon_d^a; a \in G_d, d|n \rangle / \langle \pm \epsilon_n \rangle$$

the group of the n th cyclotomic numbers.

Lemma 5 *The sequence*

$$0 \rightarrow T \rightarrow \mathcal{L}(n)/(1 - \sigma)\mathcal{L}(n) \xrightarrow{\mu} D^{(n)} \rightarrow 1 \quad (*)$$

where T is the torsion group of $\mathcal{L}(n)/(1 - \sigma)\mathcal{L}(n)$ is exact. The homomorphism μ is defined by the maps $\mu_d : G_d \rightarrow D^{(n)}$, $a \mapsto 1 - \epsilon_d^a$ for $d|n$.

→ From $(*)$ follows $\mathcal{L}(n)_+ \cong D^{(n)}$.

Let $\widehat{D}^{(n)} = D^{(n)} / \prod_{d|n, d \neq n} D^{(d)}$.

Theorem 2 *Let $\widehat{B}_d \subseteq D^{(n)}$ define a basis of $\widehat{D}^{(d)}$.*

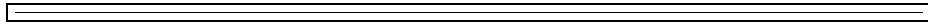
(a) $\bigcup_{d|n} \widehat{B}_d$ is a basis of $D^{(n)}$ if $4 \nmid n$.

(b) $\{1 - \epsilon_4\} \cup \bigcup_{\substack{d|n \\ d \neq 2,4}} \widehat{B}_d$ is a basis of $D^{(n)}$ if $4|n$.

Define the group of n th **cyclotomic units** by

$$C^{(n)} = D^{(n)} \cap (\mathbf{Z}[\epsilon_n]/\langle \pm\epsilon_n \rangle).$$

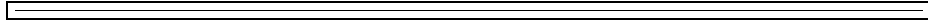
$$\text{Let } \widehat{C}^{(n)} = C^{(n)} / \prod_{d|n, d \neq n} C^{(d)}.$$



The connection between cyclotomic units and cyclotomic numbers is given by the two isomorphisms

$$\widehat{C}^{(n)} \cong \widehat{D}^{(n)} \text{ if } n \text{ is not a prime power,}$$

$$\widehat{C}^{(q)} \cong \left\langle \frac{1 - \epsilon_q^a}{1 - \epsilon_q}; a \in G_q \right\rangle \leq \widehat{D}^{(q)} \text{ if } n = q \text{ is a prime power.}$$



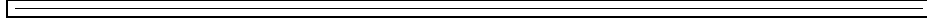
Theorem 3 If $\widehat{B}_d \subseteq C^{(n)}$ defines a basis of $\widehat{C}^{(d)}$ for $d|n$ then $B_n = \bigcup_{d|n} \widehat{B}_d$

is a basis of $C^{(n)}$.

$$\longrightarrow \bigcup_{d \in \mathbf{N}} \widehat{B}_d \text{ defines a basis of } C^{(\infty)} := \bigcup_{d \in \mathbf{N}} C^{(d)}.$$

Consider again the exact sequence

$$0 \rightarrow T \rightarrow \mathcal{L}(n)/(1 - \sigma)\mathcal{L}(n) \rightarrow D^{(n)} \rightarrow 1.$$



There are three kinds of relations in $D^{(n)}$.

Norms: $N_{\mathbf{Q}(\epsilon_n) \rightarrow \mathbf{Q}(\epsilon_d)}(1 - \epsilon_n) \in D^{(d)}$

$$\begin{aligned} \text{for instance: } (1 - \epsilon_{18})(1 - \epsilon_{18}^7)(1 - \epsilon_{18}^{13}) &= 1 - \epsilon_6 \\ &\longrightarrow \text{relations in } \mathcal{L}(n). \end{aligned}$$

Complex conjugation:

$$\begin{aligned} 1 - \epsilon_n &= -\epsilon_n \overline{1 - \epsilon_n} = -\epsilon_n(1 - \epsilon_n^{-1}) \\ &\longrightarrow \text{factoring out } (1 - \sigma)\mathcal{L}(n). \end{aligned}$$

Ennola-relations: ...

$$\longrightarrow T.$$

Ennola-relations can be constructed explicitly by means of σ -bases. We have

$$T \cong H^0(\sigma, \mathcal{L}(n)) \cong \mathbf{F}_2^{m^+(\mathcal{L}(n))}.$$

A similar construction as for the group of cyclotomic units can be done for the **Stickelberger ideal**. Let I_n the ideal generated by the Stickelberger elements

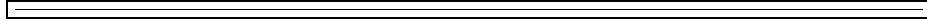
$$\theta(a) = \sum_{\tau \in G_n} \langle -a\tau/n \rangle \tau^{-1}$$

and $\omega_n = \Sigma(G_n)$ for n odd and $\omega_n = \frac{1}{2}\Sigma(G_n)$ for n even. Then we have an exact sequence

$$0 \rightarrow T \rightarrow \mathcal{L}(n)/(1 + \sigma)\mathcal{L}(n) \xrightarrow{\nu} I_n/\langle \omega_n \rangle \rightarrow 0.$$

where T is the torsion group of $\mathcal{L}(n)/(1 + \sigma)\mathcal{L}(n)$. The homomorphism ν is given by the maps

$$\nu_d : G_d \rightarrow I_n, a \mapsto \theta(an/d).$$



So with the same mechanism used for cyclotomic units we can construct bases and relations, especially Ennola-relations for I_n .